② $\lim_{\delta a \to 0} \frac{\delta F}{\delta a} = \frac{\partial F}{\delta x} = \frac{i}{\hbar}[\tilde{p}, F]$

$\Rightarrow \quad [\tilde{p}, F(\tilde{x})] = -i\hbar \frac{\partial}{\partial \tilde{x}} F$

If $F(\tilde{x}) = \tilde{x}$, $\quad [\tilde{x}, \tilde{p}] = i\hbar$.

ex. time-evolution.

$\delta A = \delta t \, [A, H]_{P.B.} \longrightarrow \delta A = \frac{i}{\hbar} \delta t \, [H, A]$

$\Rightarrow \quad \frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$: Heisenberg EOM.

Thus.

"G" is the same!

Classical $\longrightarrow$ Quantum. $\quad$ "$e^{-\frac{i}{\hbar}\alpha G}$"

Canonical transformation $\quad$ Unitary transformation.

$\left(\begin{array}{l} Q_i = q_i + \alpha \frac{\partial G}{\partial p_i} \\ P_i = p_i - \alpha \frac{\partial G}{\partial q_i} \end{array} \right) \quad \left(1 - \frac{i}{\hbar}\alpha G \right)$

(1) Rotations in C.M. and Q.M.

The trouble! : $\quad U(\alpha_1) U(\alpha_2) \neq U(\alpha_2) U(\alpha_1)$

"non-Abelian"

$$e^{-\frac{i}{\hbar}\alpha_1 G_1} e^{-\frac{i}{\hbar}\alpha_2 G_2} = \exp\left[-\frac{i}{\hbar}(\alpha_1 G_1 + \alpha_2 G_2) \right]$$

only when $[G_1, G_2] = 0$.

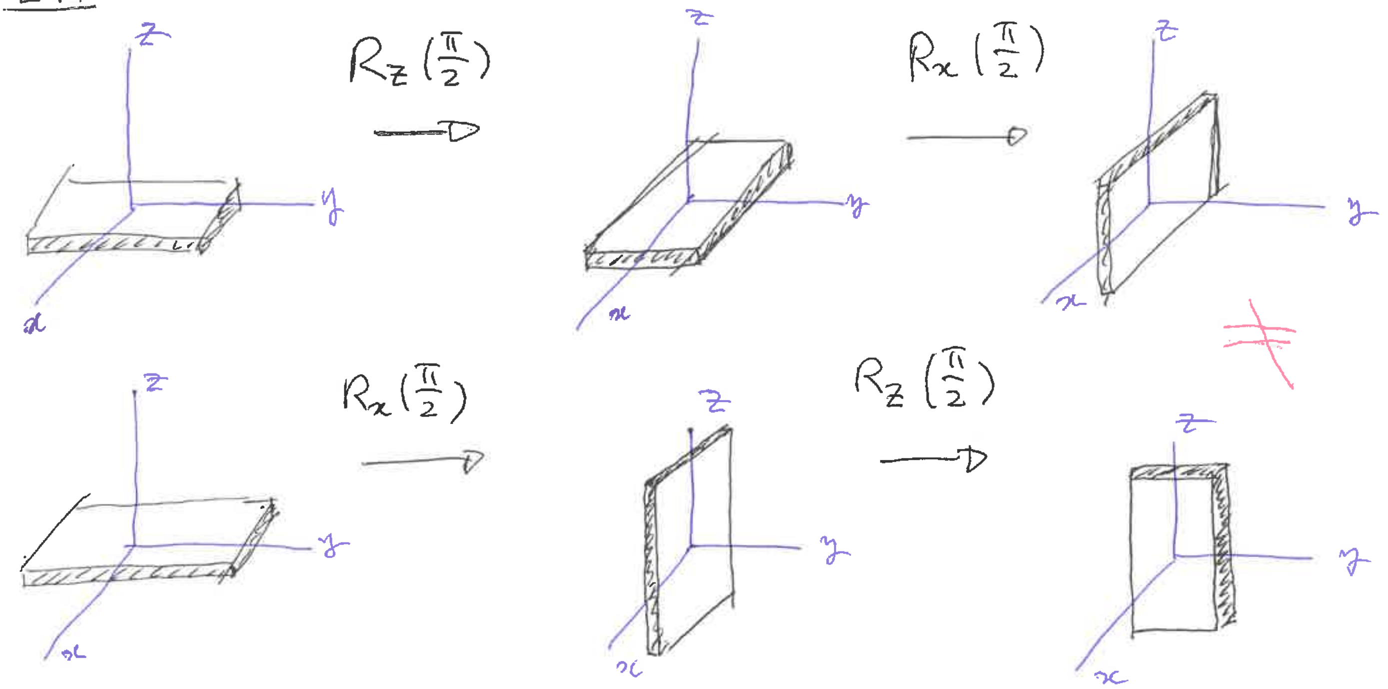
$\Leftarrow$ This is broken in general for Rotations.

(in Both of CM. and QM.).

*NOTE : We're talking about "3D" here.

!!!

Ex.



- Rotation : 3 x 3 orthogonal Matrix

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad ; \quad \boxed{RR^T = R^T R = I}$$

$$\Leftrightarrow \quad |\vec{x}'|^2 = |\vec{x}|^2$$

* NOTE

Here we consider mainly "ACTIVE" rotations.

Active : an object is rotating while coordinates are still.

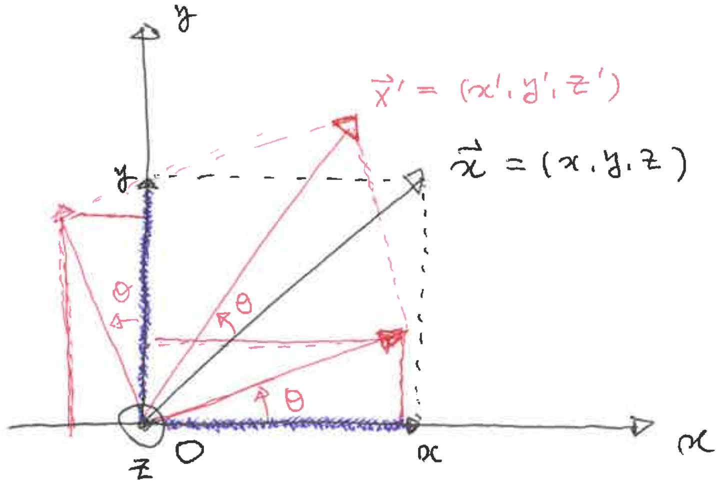
passive : coordinates are rotating while an object is still.

- How can we find R ?

① Trigonometry ← Approach I

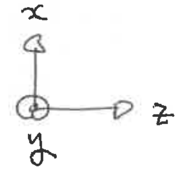
② Euler Angles

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

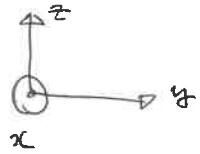


$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

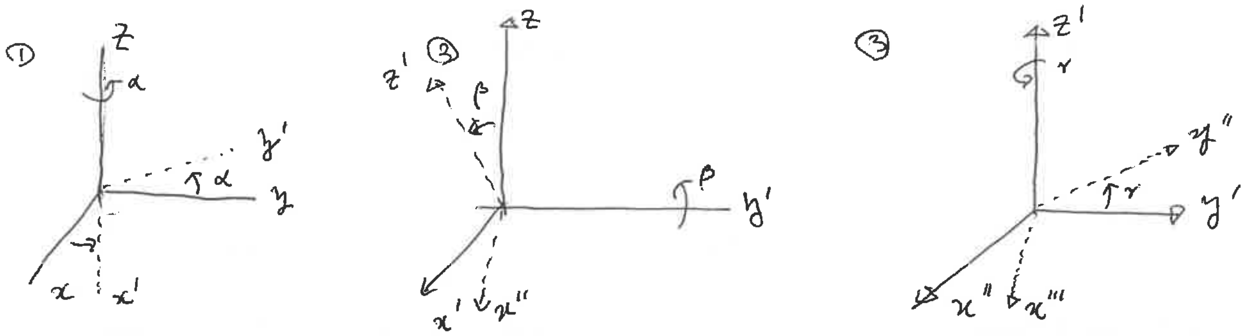


$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$



$\Rightarrow$ a general rotation matrix : Euler Angles

$$R(\alpha, \beta, \gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha) \quad (\text{body-axis rot.})$$



But, note that $R_{y'}(\beta) R_z(\alpha) = R_z(\alpha) R_y(\beta)$.

$$\Rightarrow \quad R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha).$$

Similarly,

$$R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta)$$

$$\Rightarrow \quad R(\alpha, \beta, \gamma) = R_z(\gamma)\, R_y(\beta)\, R_z(\alpha) \quad (\text{fixed-axis rot.})$$

Euler rotations

② Infinitesimal Rotations ⟸ Approach II.

$\boxed{SO(3)}$

: This is what we need to find the "Generators" of rotations.

* Group $\{g_\alpha\}$: multiplication / composition Axioms

1 Associativity : $(g_\alpha \cdot g_\beta) \cdot g_\gamma = g_\alpha \cdot (g_\beta \cdot g_\gamma)$

2. Existence of the identity I : $I \cdot g_\alpha = g_\alpha$, $g_\alpha \cdot I = g_\alpha$

3. Existence of the inverse g_α^{-1} : $g_\alpha \cdot g_\alpha^{-1} = I$, $g_\alpha^{-1} g_\alpha = I$

$\{R\}$: $SO(3)$

S: special : $\det[R] = 1$

O: orthogonal : It's an orthogonal matrix.

(3): dimensionality

~ fix an rotation axis at $\vec{\theta} = \theta \hat{n}$ to recover "Abelian"

: $R(\theta_1)\, R(\theta_2) = R(\theta_1 + \theta_2)$

↳ infinitesimal Rotation : $R \simeq I + A$

Orthogonality : $R^T R = I = (I + A^T)(I + A)$

$$= I + (A^T + A) + O(A^2)$$

$\Rightarrow$ $A = -A^T$: antisymmetric.

$\Rightarrow$ Only 3 undetermined elements.

For the special cases,

i) $\hat{n} = \hat{x}$

$\Rightarrow A^k_{ij} = -\varepsilon_{ijk}$

$$A = \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$x' = x$

$y' = y - \theta z$

$z' = z + \theta y$

ii) $\hat{n} = \hat{y}$

$$\theta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$x' = x + \theta z$

$y' = y$

$z' = z - \theta x$

iii) $\hat{n} = \hat{z}$

$$\theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x' = x - \theta y$

$y' = y + \theta x$

$z' = z$

$\hookrightarrow \vec{A} = -i\theta\vec{J}$: $[J_i, J_j] = i\varepsilon_{ijk}J_k$ $\quad\Big\|$ $(x,y,z) \equiv (1,2,3)$ NOTATION !

"Lie Algebra"

$(J_k)_{ij} = -i\varepsilon_{ijk}$ "

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

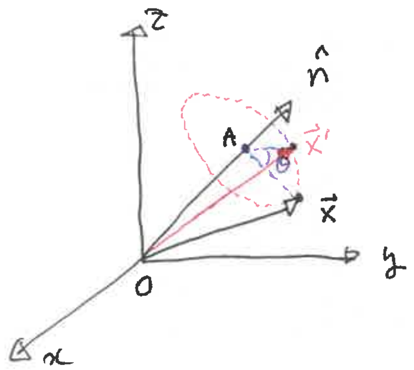
$\rightarrow$ Axis - Angle Parametrization

$$R_{\hat{n}}(\theta) = \lim_{N\to\infty}\left(I - i(\hat{n}\cdot\vec{J})\frac{\theta}{N}\right)^N = e^{-i\theta(\vec{J}\cdot\hat{n})} \quad "$$

$$= \exp\left[-i\theta(n_x J_x + n_y J_y + n_z J_z)\right]$$

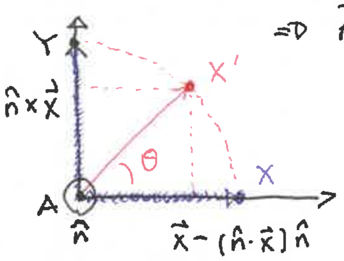
$$\Rightarrow \boxed{R_{\hat{n}}(\theta) = e^{-i\theta(\vec{J}\cdot\hat{n})}} \;\star$$

* Verification with Trigonometry.



i) $\vec{OA} = (\hat{n}\cdot\vec{x})\hat{n}$

ii)



$$\Rightarrow \vec{AX'} = \vec{AX}\cdot\cos\theta + \vec{AY}\cdot\sin\theta$$

$$= (\vec{x} - (\hat{n}\cdot\vec{x})\hat{n})\cos\theta + (\hat{n}\times\vec{x})\sin\theta$$

$$\Rightarrow \vec{x}' = (\hat{n}\cdot\vec{x})\hat{n} + (\vec{x}-(\hat{n}\cdot\vec{x})\hat{n})\cos\theta + (\hat{n}\times\vec{x})\sin\theta$$

For $\theta \ll 1$, $\quad \vec{x}' \cong \vec{x} + \theta(\hat{n}\times\vec{x})$

$$\hookrightarrow \vec{x}' \simeq [I - i\theta(\vec{J}\cdot\hat{n})]\vec{x}$$

$$\Rightarrow (\vec{J}\cdot\hat{n})\vec{x} = i(\hat{n}\times\vec{x}) = i\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$(\cdot)_k \Rightarrow \varepsilon_{ijk} n_i x_j$